Probability Theory

Introduction

Probability theory is an area of mathematics that deals with random events where we calculate how likely it is for an event to occur. It is used widely in the world of statistics because it can be used to make predictions or analyse the most likely events to occur in an experiment and compare these results to theoretical predictions. Probability Theory can be traced back to peoples' attempts to analyse games of chance and dice (before the mathematics of probability was invented).

Aim of Workshop:

The aim of this workshop is to introduce students to the basics of Probability Theory and its applications through the Monty Hall Problem. The Monty Hall problem is a famous problem based on the American television show "Let's Make a Deal" and is named after the show's host – Monty Hall.

Learning Outcomes

By the end of this workshop students should be able to:

- Discuss, in their own words, the concept of probability and sample spaces
- Describe the Monty Hall problem
- Recognise the applications of Probability Theory.

Keywords

Sample space

The list of all possible outcomes from some experiment E.g. the sample space for rolling a six sided dice is given as: S = {1, 2, 3, 4, 5, 6}.

Probability Theory: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Play the Monty Hall Game**	– Ask for 3 students to volunteer to play "Let's Make a Deal" – Ask each student to give the best definition of probability that they can – Allow the rest of the class to decide which student gave the best definition – The student who gets the most votes will play the game (See **Appendix – Note 1** for the game show rules)
10 mins (00:15)	**Probability Theory definitions**	– Introduce the topic of the workshop and give some definitions required for the lesson, such as sample space, event space etc. (See **Appendix – Note 2**) – Students should attempt **Activity Sheet 1**
10 mins (00:25)	**Play the 3 card version of the Monty Hall game**	– Allow the students to play the game in pairs using 3 cards. – Ask: "Do we think you win more often if you switch or stay when offered the chance? Or will it make any difference at all?" – Students should try both strategies and experiment as to how often they win in e.g. 10 plays when they stay every time and then when you switch every time.
10 mins (00:35)	**Solve the Monty Hall 3 card problem**	– Ask students to attempt **Activity Sheet 2** (Remind students to keep in mind the basic concepts that they have seen before)
10 mins (00:45)	**Explain the solution**	– Discuss the solutions to the Monty Hall problem and why discuss switching is the best thing to do over many times playing the game (see **Appendix – Note 3**)
10 mins (00:55)	**The 4 card Monty Hall problem**	– Let the student play the 4 card version of the game to reinforce the idea that swapping is the better option. – Students should attempt **Activity Sheet 3**
15 mins (01:10)	**N door Monty Hall problem**	– Ask students to attempt **Activity Sheet 4** and see if the can work out what happens to the chances of winning when you switch for a large amount of doors. – Discuss some of the applications of probability (see **Appendix – Note 4**)

Probability Theory Appendix

Note 1 – Let's Make a Deal/ Monty Hall problem Rules

- There are three doors and behind one of the doors is a brand new car (or other desirable item) and behind the other two doors are goats (or other nonsense prize)
- The contestant is given the choice to choose one of the three doors.
- After the contestant chooses a door, the host will then reveal what's behind one of the other two remaining doors, but he will only ever reveal a goat and never the car (the host always knows where the car is).

  For example, if the car is behind door 2 and the contestant chose door 1, then the host will open door 3 (revealing one of the goats).

  If the contestant initially chose (door 2) that contains the car, then the host will open either of the other two doors (door 1 or door 3) revealing one of the goats
- After the host reveals a goat, the host then gives the contestant the option of staying with their original door or switching to the one remaining door.
- The big question is: Is it in our best interest to switch or stay?

Note 2 – Probability Theory Definitions

The main definition we need for working out probabilities is also given at the top of Activity Sheet 1:

$$\text{Probability of an Event} = \frac{\text{Total number of successful outcomes}}{\text{Total number of possible outcomes}}$$

For example, the probability of rolling an even number on a die is worked out by finding the individual components of the definition:

Total number of successful outcomes = {2, 4, 6} = 3 possible outcomes

Total number of possible outcomes = {1, 2, 3, 4, 5, 6} = 6 total outcomes

Therefore:

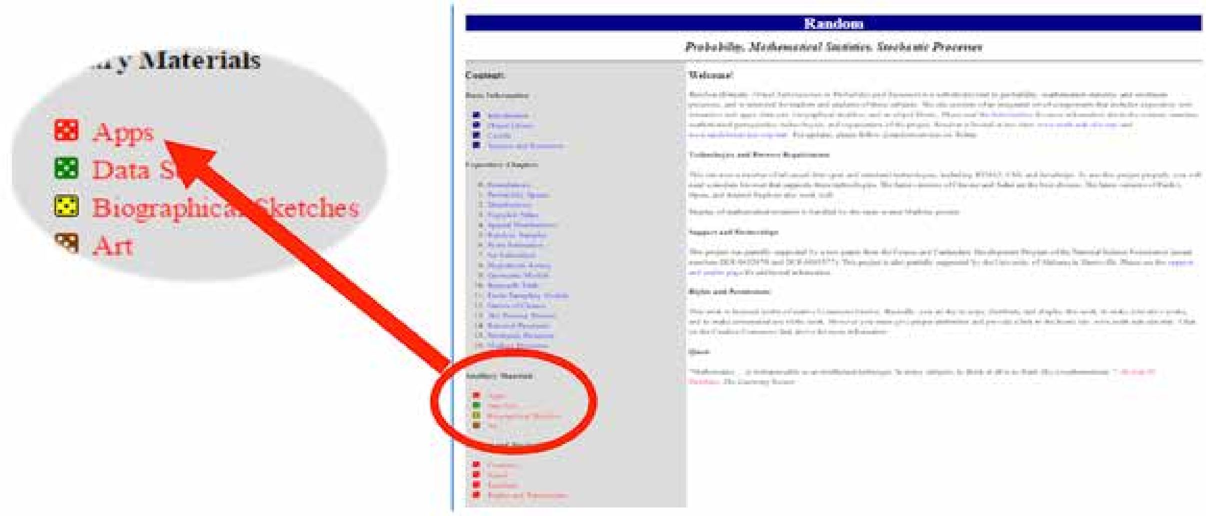
$$\text{Probability of rolling an even number} = \frac{3}{6} = \frac{1}{2} = 0.5$$

Additionally, we can make use of the rule that when we run an experiment the sum of the probabilities of each outcome occurring must add up to 1.

In the above example the probability of rolling an even number was ½ and similarly the probability of the other outcome: rolling an odd number is also ½, and the sum of these probabilities is 1.

Note 3 – Monty Hall Problem

The following is a website that provides a number of statistics apps and simulations:
http://www.math.uah.edu/stat/



Upon opening the website, choose the applets section on the left–hand side of the screen:

The list of apps that opens come from a large number of different areas of statistics but the one we are interested is under the section ***"Games of Chance" / "Monty Hall Game" or "Monty Hall Experiment"***

The **"Monty Hall Game"** is a way to play the Monty Hall Game online. The applet comes with a full description of the game and instructions on how to use the applet. This app will allow you to run individual games of Let's Make a Deal and will work well if you do not have the ability to run the game physically in the classroom.

The **"Monty Hall Experiment"** will allow you to run the game many times with the option of setting that the player always stays or always swaps doors. (Please see the applet description to get more details on the various labels on screen.)

There are also resources available on:
https://www.youtube.com/watch?annotation_id=annotation_3286749133&feature=iv&src_vid=4Lb–6rxZxx0&v=ugbWqWCcxrg

Note 4: Applications of Probability Theory

Weather Prediction:

- What is the probability that it will rain tomorrow?
- What are the chances that an earthquake will cause a tsunami?

Sport:

- How likely is each team in the next World Cup to win?
- How do bookies calculate odds on various sports games?

Entertainment:

- How likely is the next Leonardo di Caprio movie to win an Oscar?
- How likely is it for an Irish language movie to win at the Cannes film festival?
- What is the probability that U2 will play in Croke Park next year?

Note 5: Activity Sheet 4 Solutions

(N–Door) Monty Hall Problem

Problem

In this case there are initially N doors to pick from and the host opens N–2 doors (one chosen by the contestant). Everything else in the problem remains unchanged. N can be any number – in this case how many doors there are initially, for example N=12.

Solution

Answer: The contestant should switch because P (win if stay) = $\boxed{\frac{1}{N}}$

& P (win if switch) = $\boxed{\frac{N-1}{N}}$

Probability Theory – Activity Sheet 1

1 What are all the outcomes of throwing an 8–sided die? This is known as a sample space.

2. What is the probability of landing on an even number on an 8–sided die?

3 What is the probability of rolling a 5 or greater on a 4–sided die?

4. In a certain game it counts as a win if a contestant rolls higher than an 8 on a 12–sided die. What is the probability of winning the game?

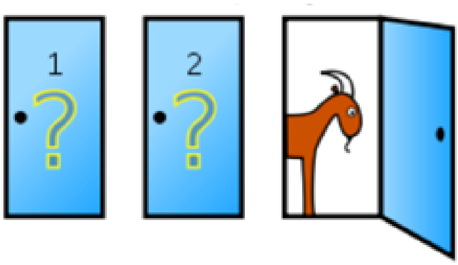
5. What is the probability of the sun rising tomorrow morning?

Probability Theory – Activity Sheet 2

(3–Door) Monty Hall Problem

Problem

- You're given the choice of three doors: behind one door is a car; behind the other two (losing) doors are goats.
- You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. The host always picks a (losing) door with no prize behind it and always knows where the car is.
- The host will always offer the contestant a chance to switch after opening one of the two remaining doors.



Q. Are you more likely to win by switching?

Solution to Monty Hall 3–card Problem

Q. What is the sample space when you switch? When you stay?

Let's play the game with three cards: one black (winning) card and two red (losing) cards. Let Card 1 be the winning card.

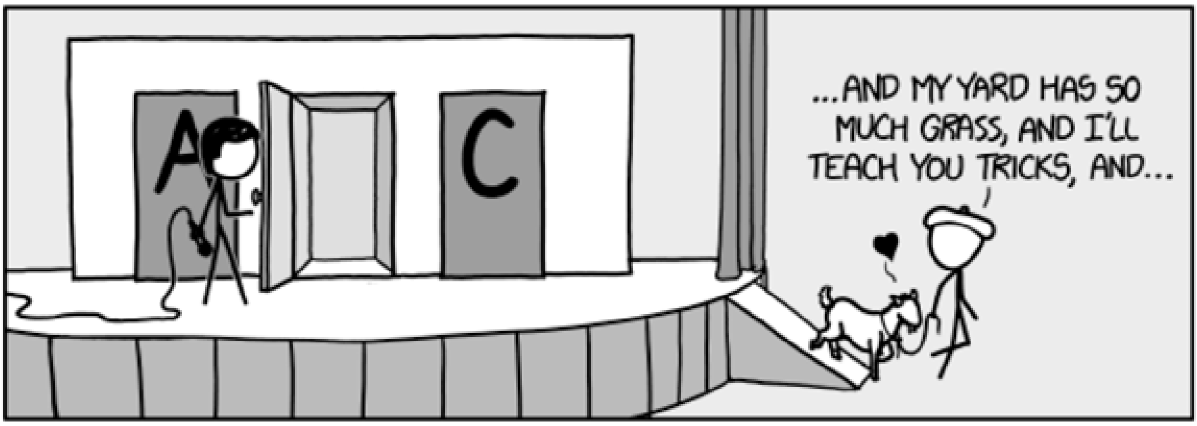
For example, if a contestant chooses card 1 and then decides to not switch when offered, then they will win the car. Enter "Win" or "Lose" for each option in the below table:

CONTESTANT'S INITIAL CHOICE OF CARD	STAY	SWITCH
Card 1		
Card 1		
Card 2		
Card 2		
Card 3		
Card 3		
P(Win)=		

The solution will be similar if card 2 or card 3 is the winning card. The probability of winning, when switching/staying remains the same regardless of which card is the winning card.
Now try entering the outcomes in the condensed table below:

CONTESTANT'S **INITIAL** CHOICE OF CARD	STAY	SWITCH
Card 1		
Card 2		
Card 3		
P(Win)=		

Though nobody said you didn't want to win the goat...



Credit: http://xkcd.com/1282/

http://imgs.xkcd.com/comics/monty_hall.png

Probability Theory – Activity Sheet 3

(4–Card) Monty Hall Problem

Problem

In the 4–card Monty Hall problem, you're given the choice of **four cards**. One card is black (the winning card); the other 3 cards are red (losing cards). You pick a card, say No. 1, and the host, who knows what the cards are, **picks two other cards**, say No. 3 and No. 4, which are both red (losing cards). He then says to you, "Do you want to pick card No. 2?"

Q. Is it to your advantage to switch your choice?

Fill in the following table just like in the last activity sheet. Just keep in mind that now you have 4 cards rather than 3 as before.

Please note the winning card is card one.

CONTESTANT'S **INITIAL** CHOICE OF CARD	STAY	SWITCH
1		
2		
3		
4		
P(Win)=		

Solution

Answer: The contestant should switch because P (win if stay) = ☐

and P (win if switch) = ☐

Probability Theory – Activity Sheet 4

(N–Door) Monty Hall Problem

Problem

In this case there are initially N doors to pick from and everything else in the problem remains unchanged. Can you come up with a general formula to help the contestant decide if they should switch or stay?

Q. Is it to the advantage of the contestant to switch their choice?

P (win if stay) =

and P (win if switch) =

The contestant should always

because

Hint:

As we know the probabilities of winning the 3–Door, and 4–Door, try putting in N=3, and N=4 into the N–Door solution, and seeing if they match the previous probabilities.